MATH5011 Exercise 1

(1) Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of measurable sets in $(X, \mathcal{M})$. Let

$$A = \{x \in X : \ x \in A_k \text{ for infinitely many } k\} \ ,$$

and

$$B = \{x \in X : x \in A_k \text{ for all except finitely many } k\} \ .$$

Show that A and B are measurable.

(2) Let $\Psi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous. Show that $\Psi(f, g)$ are measurable for any measurable functions f, g. This result contains Proposition 1.3 as a special case.

(3) Show that $f : X \to \overline{\mathbb{R}}$ is measurable if and only if $f^{-1}([a, b])$ is measurable for all $a, b \in \overline{\mathbb{R}}$.

(4) Let $f : \ X \times [a, b] \to \mathbb{R}$ satisfy (a) for each x, $y \mapsto f(x, y)$ is Riemann integrable, and (b) for each y, $x \mapsto f(x, y)$ is measurable with respect to some σ-algebra $\mathcal{M}$ on X. Show that the function

$$F(x) = \int_a^b f(x, y)dy$$

is measurable with respect to $\mathcal{M}$.

(5) Let f, g, $f_k, k \geq 1$, be measurable functions from X to $\overline{\mathbb{R}}$.

1. Show that $\{x : f(x) < g(x)\}$ and $\{x : f(x) = g(x)\}$ are measurable sets.

2. Show that $\{x : \lim_{k\to\infty} f_k(x) \text{ exists and is finite}\}$ is measurable.

(6) There are two conditions (i) and (ii) in the definition of a measure μ on $(X, \mathcal{M})$. Show that (i) can be replaced by the "nontriviality condition": There exists some $E \in \mathcal{M}$ with $\mu(E) < \infty$.

(7) Let $\{A_k\}$ be measurable and $\displaystyle\sum_{k=1}^{\infty} \mu(A_k) < \infty$ and

$$A = \{x \in X : x \in A_k \text{ for infinitely many } k\}.$$

From (1) we know that A is measurable. Show that $\mu(A) = 0$.

(8) Let B be the set defined in (1). Let μ be a measure on $(X, \mathcal{M})$. Show that

$$\mu(B) \leq \liminf_{k\to\infty} \mu(A_k) \ .$$

(9) Here we review Riemann integral. This is an optional exercise. Let f be a bounded function defined on $[a, b], a, b \in \mathbb{R}$. Given any partition $P = \big\{a = x_0 < x_1 < \cdots < x_n = b\big\}$ on $[a, b]$ and tags $z_j \in [x_j, x_{j+1}]$, there corresponds a *Riemann sum* of f given by $R(f, P, \mathbf{z}) = \sum_{j=0}^{n-1} f(z_j)(x_{j+1} - x_j)$. The function f is called *Riemann integrable* with integral L if for every $\varepsilon > 0$ there exists some δ such that

$$\big|R(f, P, \mathbf{z}) - L\big| < \varepsilon,$$

whenever $\|P\| < \delta$ and $\mathbf{z}$ is any tag on P. (Here $\|P\| = \max_{j=0}^{n-1} |x_{j+1} - x_j|$ is the length of the partition.) Show that

(a) For any partition P, define its *Darboux upper* and *lower sums* by

$$\overline{R}(f, P) = \sum_j \sup\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j),$$

and

$$\underline{R}(f,P) = \sum_j \inf\big\{f(x) : x \in [x_j, x_{j+1}]\big\}(x_{j+1} - x_j)$$

respectively. Show that for any sequence of partitions $\{P_n\}$ satisfying $\|P_n\| \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \overline{R}(f,P_n)$ and $\lim_{n\to\infty} \underline{R}(f,P_n)$ exist.

(b) $\{P_n\}$ as above. Show that f is Riemann integrable if and only if

$$\lim_{n\to\infty} \overline{R}(f,P_n) = \lim_{n\to\infty} \underline{R}(f,P_n) = L.$$

(c) A set E in $[a,b]$ is called *of measure zero* if for every $\varepsilon > 0$, there exists a countable subintervals J_n satisfying $\sum_n |J_n| < \varepsilon$ such that $E \subset \bigcup_n J_n$. Prove Lebsegue's theorem which asserts that f is Riemann integrable if and only if the set consisting of all discontinuity points of f is a set of measure zero. Google for help if necessary.